MATH 20D Spring 2023 Lecture 3.

Linear IVP's, the logistics equations, and 1st order IVP's.

Outline

Contents

Definition

An **n-th order linear initial value problem (IVP)** is an n-th order linear ODE

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

together with a family of initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

such that $x_0, y_0, y_1, \ldots, y_{n-1}$ are fixed constants and the functions

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Theorem

An n-th order linear IVP has a unique solution $y_{\text{sol}}(t)$ on any interval $I \subseteq \mathbb{R}$ on which the functions $a_{n-1}(t), \ldots, a_0(t)$, and $g(t)$ are all continuous.

Example

Show that the first order linear IVP

$$y'(t) - y(t) = 0, \qquad y(0) = 1$$

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Existence: Recall the familiar exponential function

$$y\colon \mathbb{R} \to \mathbb{R}_{>0}, \qquad y(x) = e^x.$$

One way to define e^x is by the **power series** $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

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- The series converges absolutely for all $x \in \mathbb{R}$ (ratio test).
- However to justify the formal calculation

$$\frac{d}{dx}(e^x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = e^x$$

requires material from MATH 140B. (uniform convergence).

Alternative proof for the existence of e^x.

- Define $\log(\cdot)\colon \mathbb{R}_{>0} \to \mathbb{R}$ by the definite integral

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- Since $\log'(x)$ is never zero $\exp(x)$ is everywhere differentiable. So if $y = \log(x)$ then $x = \exp(y)$ and
$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{\frac{d}{dx}(\log(x))} = \frac{1}{1/x} = x = \exp(y).$$

- One advantage of the power series definition

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1}$$

is that it makes sense even if $s, t \in \mathbb{R}$ and

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Example

Let $\omega \in \mathbb{R}_{>0}$. In case (a) and (b) below, write down the solution to the IVP

$$y''(t) + \omega^2 y(t) = 0$$

subject to the initial conditions

(a) $y(0) = 1$ and $y'(0) = 0$.

(b) $y(0) = 1$ and $y'(0) = i\omega$.

Contents

A population $P(t)$ has carrying capacity $K > 0$ and growth rate $r > 0$.

- One model for the change in population is the **Logistics Equation**

$$\frac{dP}{dt} = rP(1 - \frac{P}{K})$$

A population $P(t)$ has carrying capacity $K > 0$ and growth rate $r > 0$.

- One model for the change in population is the **Logistics Equation**

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}) \tag{2}$$

Example

Suppose $C > 0$ is constant and $P \colon \mathbb{R} \to \mathbb{R}$ satisfies

$$P(t) = Ce^{rt}(1 - P(t)/K).$$

- Use implicit differentiation to show that $P(t)$ solves the logistics equation.
- Show that if $P_0 \notin \{0, K\}$ denotes the initial population then the function

$$P \colon \mathbb{R} \to \mathbb{R}, \qquad P(t) = \frac{P_0 K e^{rt}}{K + P_0(e^{rt} - 1)}$$

solves (2) and satisfies $P(0) = P_0$.

Question

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Theorem

Consider a first order IVP of the form

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0 \tag{3}$$

where x_0, y_0 *are constants and* $f(x, y)$ *is a function. Suppose there exist a rectangle*

$$R = \{(x, y) \in \mathbb{R}^2 : a < x < b, \quad c < y < d\}$$

such that $(x_0, y_0) \in R$, $f(x, y)$ *continuous of* R, *and* $\partial f / \partial y$ *is continuous on* R.

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$$\{x \in \mathbb{R} \colon x_0 - \delta < x < x_0 + \delta\}.$$

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We can use this to give an **affirmative** answer to the question above.

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Let C be constant and consider the non-linear ordinary differential equation

$$y\frac{dy}{dx} - 4x = 0$$

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$$y\frac{dy}{dx} - 4x = 0 \tag{4}$$

- Show if $\delta > 0$ then the functions

  $$y_1 : (-\delta, \delta) \to \mathbb{R}, \quad y_1(x) = 2x \qquad \text{and} \qquad y_2 : (-\delta, \delta) \to \mathbb{R}, \quad y_2(x) = -2x,$$

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  are both solutions to (4) satisfying $y_1(0) = 0 = y_2(0)$.
- Explain why this is consistent with the theorem on the previous slide.